# SYMMETRIC ACHROMATIC BEAM TRANSPORT SYSTEMS

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Although these systems have been studied elsewhere it may be useful to summarize some formulas for quick reference.

With the dispersive displacement-slope vector written as ($x$, $x'$, $\Delta p/p$) the transfer matrix is of the form

$$M = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where

$$\begin{vmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{vmatrix} = ad - bc = 1.$$

Several general relations are useful:

1) The inverse matrix is

$$M^{-1} = \begin{pmatrix} d & -b & bf-de \\ -c & a & -af+ce \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

2) If the polarity of all the bending magnets is reversed and the orbit is bent in the opposite direction $x$, hence also $x'$, changes sign and the transfer matrix is

$$M^R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & -e \\ c & d & -f \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

3) If $M$ is the transfer matrix of the first half of a symmetric system that of the second half is

$$M^S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d & b & bf-de \\ c & a & af-ce \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

4) If the second half of a symmetric system has the polarity of all bending magnets reversed the transfer matrix is

$$M^{SR} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} M^S \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d & b & de-bf \\ c & a & ce-af \\ 0 & 0 & 1 \end{pmatrix} \qquad (5)$$

For specific beam transport elements the transfer matrices are

(a) Gradient bending magnet "G" with ends perpendicular to beam

$$\left\{ \begin{aligned} M_x &= \begin{pmatrix} \cos\phi & \frac{\rho}{\sqrt{1-n}}\sin\phi & \frac{\rho}{1-n}(1-\cos\phi) \\ -\frac{\sqrt{1-n}}{\rho}\sin\phi & \cos\phi & \frac{1}{\sqrt{1-n}}\sin\phi \\ 0 & 0 & 1 \end{pmatrix} \\ M_y &= \begin{pmatrix} \cos\psi & \frac{\rho}{\sqrt{n}}\sin\psi \\ -\frac{\sqrt{n}}{\rho}\sin\psi & \cos\psi \end{pmatrix} \end{aligned} \right. \qquad (6)$$

where $\phi \equiv \sqrt{1-n}\,\Theta$, $\psi \equiv \sqrt{n}\,\Theta$, $\Theta$ is the bending angle, $\rho$ is the radius of curvature, and $n \equiv -\rho^2\left(B'/B\rho\right) = -\rho^2\left[\text{field gradient}/(B\rho)\text{ of the particle}\right]$ is the conventional field gradient index. Also, since there is no dispersion in the y-direction, $M_y$ is written as a 2×2 matrix for the displacement-slope vector $(y,y')$.

For bending magnet with uniform field and ends perpendicular to beam the transfer matrices are obtained from Eq. (6) by simply putting $n = 0$.

(b) Bending magnet "B" with uniform field and parallel ends.

$$\left\{ \begin{aligned} M_x &= \begin{pmatrix} 1 & \rho\sin\Theta & 2\rho\sin^2\Theta/2 \\ 0 & 1 & 2\tan\Theta/2 \\ 0 & 0 & 1 \end{pmatrix} \\ M_y &= \begin{pmatrix} 1-\Theta\tan\Theta/2 & \rho\Theta \\ -1/\rho(\tan\Theta/2)(2-\Theta\tan\Theta/2) & 1-\Theta\tan\Theta/2 \end{pmatrix} \end{aligned} \right. \qquad (7)$$

where $\rho$ is the radius of curvature and $\Theta$ is the bending angle and where, since there is no dispersion in the y-direction, $M_y$ is written as a 2×2 matrix for the displacement-slope vector $(y,y')$.

(c) Quadrupole magnet "Q"

$$M_x = \begin{pmatrix} \cos\omega\ell & 1/\omega\sin\omega\ell & 0 \\ -\omega\sin\omega\ell & \cos\omega\ell & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_y = \begin{pmatrix} \cosh\omega\ell & 1/\omega\sinh\omega\ell \\ \omega\sinh\omega\ell & \cosh\omega\ell \end{pmatrix} \qquad (8)$$

where $\omega^2 \equiv B'/B\rho$ = field gradient of "Q"/[$(B\rho)$ of the particle] and $\ell$ = length of "Q". When the polarity of "Q" is reversed the 2×2 part of $M_x$ and $M_y$ will simply be interchanged.

(d) Edge angle of bending magnet "E"

$$M_x = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\tan\alpha}{\rho} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_y = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\alpha}{\rho} & 1 \end{pmatrix} \qquad (9)$$

where $\alpha$ is the edge angle taken to be positive when the edge is horizontally defocusing.

(e) Drift space "D"

$$\left\{ \begin{array}{l} M_x = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \\ M_y = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \end{array} \right. \qquad (10)$$

where s is the length of the drift space.

Since "Q" "D" and "E" are all nondispersive we can consider the symmetric achromatic system to begin and end with "B". There are two general types of systems:

1. Bending system

Let the desired bending angle be $2\Theta$. Then the first half of the system will bend the beam by angle $\Theta$. For achromatism the (23) element, namely f, of the transfer matrix of the first half must vanish. This condition clearly cannot be satisfied by a "B" alone or a "B" followed by a "D". The simplest possible combination is "BDQ". However to get some flexibility we will consider the combination "BDQD" as the first half of the system. The x-transfer matrix of the first half is, therefore,

$$M_x = \begin{pmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \omega\ell & 1/\omega \sin \omega\ell & 0 \\ -\omega \sin \omega\ell & \cos \omega\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \tag{11}$$

$$\begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \rho \sin \Theta & 2\rho \sin^2\Theta/2 \\ 0 & 1 & 2\tan\Theta/2 \\ 0 & 0 & 1 \end{pmatrix}$$

The condition of achromaticity, namely the (23) element of $M_x$ must vanish, then, gives

$$\text{Cos } \omega\ell = \omega(s + \frac{\rho}{2} \sin \Theta) \equiv A. \tag{12}$$

When this condition is satisfied we have

$$M_x = \begin{pmatrix} \frac{A-R}{\sqrt{1+A^2}} & \frac{1}{\omega}\left[\frac{(A - S)(A - R)}{\sqrt{1 + A^2}} + \sqrt{1 + A^2}\right] & \frac{2}{\omega}\sqrt{1 + A^2} \tan \frac{\Theta}{2} \\ -\omega \frac{1}{\sqrt{1 + A^2}} & -\frac{A - S}{\sqrt{1 + A^2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

where

$A \equiv \omega(s + \rho/2 \sin \Theta)$

$R \equiv \omega r$

$S \equiv \omega s$

and where the (13) term gives the displacement dispersion at the middle of the system, namely

$$x = \left(\frac{2}{\omega} \sqrt{1 + A^2} \tan \frac{\Theta}{2}\right)\frac{\Delta p}{p} \tag{14}$$

Both the (13) and the (23) terms of the transfer matrix for the whole system $M_x^S M_x$ will vanish and the optical part (the 2×2 part) of $M_x^S M_x$ becomes

$$M_x^S M_x = \begin{pmatrix} 1 & \frac{1}{\omega}\left[A - S + \frac{1 - A^2}{A - R}\right] \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2\omega \frac{A-R}{1+A^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\omega}\left[A-S + \frac{1-A^2}{A-R}\right] \\ 0 & 1 \end{pmatrix} \tag{15}$$

showing that optically in the horizontal direction the system is equivalent to a thick lens with principal planes at distances $\pm 1/\omega \left[A - S + \frac{1 - A^2}{A - R}\right]$ from the ends and (focal length)$^{-1}$ equal to $-2\omega \frac{A - R}{1 + A^2}$ .

There is no particular simple way of exhibiting the vertical optics of $M_y$ of the system. One will just have to multiply the matrices involved directly.

The total length of the system is clearly

$$L = 2(\rho\Theta + s + \ell + r). \tag{16}$$

2. Displacing system

Again, considering the combination "BDQD" as the first half of the system we see that the x-transfer matrix of the first half is, again, given by Eq. (11). The condition of achromaticity is, however, now that the (13) element of $M_x$ must vanish. This gives

$$\tan \omega \ell = - \frac{A + R}{1 - AR} \tag{17}$$

When this condition is satisfied we have

$$M_x = \begin{pmatrix} \sqrt{\frac{1 + R^2}{1 + A^2}} & \frac{1}{\omega}(A - S)\sqrt{\frac{1 + R^2}{1 + A^2}} & 0 \\ \omega \frac{A + R}{\sqrt{(1+ A^2)(1+R^2)}} & \frac{(A - S)(A + R)}{\sqrt{(1 + A^2)(1 + R^2)}} + \sqrt{\frac{1+A^2}{1+R^2}} & 2\sqrt{\frac{1+A^2}{1+R^2}} \tan\frac{\Theta}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

where, again,

$$A \equiv \omega(s + \rho/2 \sin \Theta)$$

$$R \equiv \omega r$$

$$S \equiv \omega s$$

The (23) term gives the angle dispersion at the middle of the system, namely

$$x' = \left( 2\sqrt{\frac{1+A^2}{1+R^2}} \tan \frac{\Theta}{2} \right)\frac{\Delta p}{p} \tag{19}$$

The optical part of the transfer matrix $M_x^{SR} M_x$ becomes

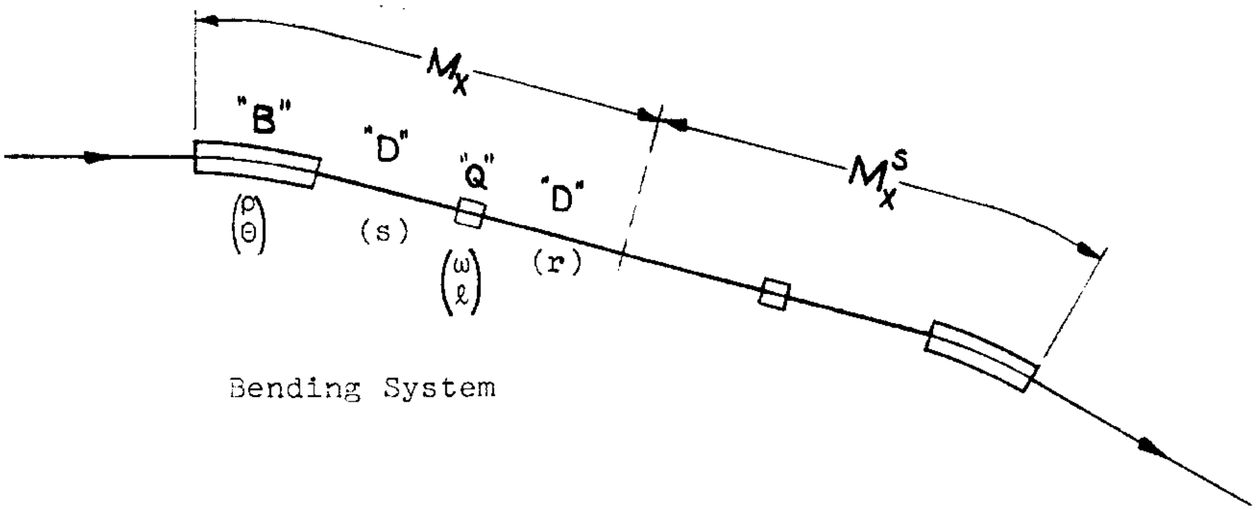
$$M_x^{SR} M_x = \begin{pmatrix} 1 & \frac{1}{\omega}(A - S) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2\omega \frac{A+R}{1+A^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\omega}(A-S) \\ 0 & 1 \end{pmatrix} \tag{20}$$

indicating that optically ω the horizontal direction the system is equivalent to a thick lens with principal planes at distances

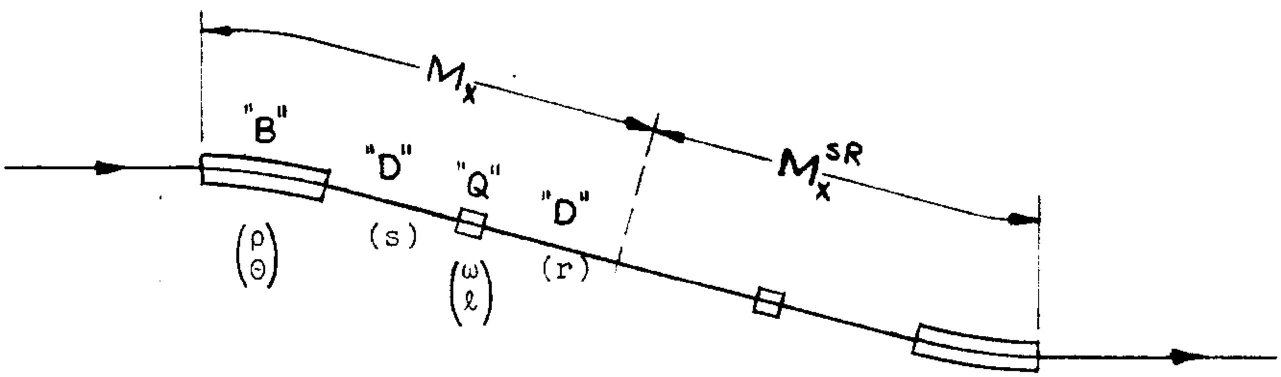
$\pm \frac{1}{\omega}$ (A-S) from the ends and (focal length)$^{-1}$ equal to $2\omega \frac{A+R}{1+A^2}$ . The total length of the system is, again, given by Eq. (16) and the total displacement of the beam is approximately $(\rho\Theta + 2s + 2\ell + 2r)\sin\Theta$.

Also, here, there is no particular simple way of exhibiting the y-optics of the system. One will just have to multiply the matrices involved directly.

The two systems are shown in the following diagrams:




Bending System




Displacing System